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ON PACKING DESIGNS WITH BLOCK SIZE 5 AND INDEX 4

Ahmed M. ASSAF* and Alan HARTMAN

IBM Israel—Science and Technology, Technion City, Haifa 32000, Israel

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A (v, k, λ) packing design of order v, block size k, and index λ is a collection of k-element subsets, called blocks, of a v-set, V, such that every 2-subset of V occurs in at most λ blocks. The packing problem is to determine the maximum number of blocks in a packing design. In this paper we solve the packing problem with $k = 5$, $\lambda = 4$, and all positive integers v.

1. Introduction

A (v, k, λ) *packing design* of *order* v, *block size* k, and *index* λ is a collection, β, of k-element subsets, called *blocks*, of a v-set, V, such that every 2-subset of V occurs in at most λ blocks.

Let $\sigma(v, k, \lambda)$ denote the maximum number of blocks in a (v, k, λ) packing design. A (v, k, λ) *packing design* with $|\beta| = \sigma(v, k, \lambda)$ will be called a *maximum* packing design. The function $\sigma(v, k, 1)$ is of importance in coding theory since the block incidence vectors of a $(v, k, 1)$ packing design form the codewords of a binary code of length v minimum distance $2(k-1)$ and constant weight k. Thus $\sigma(v, k, 1)$ is the maximum number of codewords in such a code.

Schoenheim [11] has shown that

$$\sigma(v, k, \lambda) \leqslant \left[\frac{v}{k}\left[\frac{v-1}{k-1}\right]\right] = \psi(v, k, \lambda)$$

where $[x]$ is the largest integer satisfying $[x] \leqslant x$. Hanani [7] has sharpened this bound in certain cases by proving the following result.

Theorem 1.1. *If* $\lambda(v-1) \equiv 0 \pmod{k-1}$ *and* $\lambda v(v-1)/(k-1) \equiv 1 \pmod{k}$ *then* $\sigma(v, k, \lambda) \leqslant \psi(v, k, \lambda) - 1$.

Other upper bounds on the function $\sigma(v, k, 1)$ have been given by Johnson [9] and Best et al. [2]. Lower bounds on $\sigma(v, k, \lambda)$ are generally given by construction of (v, k, λ) packings.

The value of $\sigma(v, 3, \lambda)$ for all v and λ has been determined by Schoenheim [11], and Hanani [7]. The value of $\sigma(v, 4, 1)$ has been determined for all v by

* Department of Algebra, Combinatorics and Analysis, Auburn University, Auburn Ala. 36849, USA.

Brouwer [5]; and the value of $\sigma(v, 4, \lambda)$ for all v and $\lambda > 1$ is given by Billington, Stanton and Stinson [4], and the authors [1, 8].

In order to state the results known about $\sigma(v, 5, \lambda)$ we need the following definition. A *balanced incomplete block design*, $\mathrm{B}(v, k, \lambda)$ is a (v, k, λ) packing design, where every 2-subset of points is contained in precisely λ blocks. If a $\mathrm{B}(v, k, \lambda)$ exists then it is clear that $\sigma(v, k, \lambda) = \lambda v(v-1)/k(k-1) = \psi(v, k, \lambda)$, and Hanani [7] has proved the following existence theorem for $\mathrm{B}(v, 5, \lambda)$.

Theorem 1.2. *Necessary and sufficient conditions for the existence of a* $\mathrm{B}(v, 5, \lambda)$ *are that* $\lambda(v-1) \equiv 0 \pmod 4$ *and* $\lambda v(v-1) \equiv 0 \pmod{20}$ *and* $(v, \lambda) \neq (15, 2)$.

This theorem implies, that $\sigma(v, 5, 1) = \psi(v, 5, 1)$ for all $v = 0, 1, 4, 5 \pmod{20}$, by taking a balanced incomplete block design when $v \equiv 1, 5 \pmod{20}$. For $v \equiv 0, 4 \pmod{20}$ one can construct a $\mathrm{B}(v+1, 5, 1)$ design, and delete a point and all the blocks containing it to give a packing design with $\psi(v, 5, 1)$ blocks. Best et al. [2] give a table of the values of $\sigma(v, 5, 1)$ for all $v \leq 24$, and they claim that other values of this function are known.

In this paper we are interested in determining the values of $\sigma(v, 5, 4)$. Here Theorem 1.2 implies that $\sigma(v, 5, 4) = \psi(v, 5, 4)$ for all $v \equiv 0, 1 \pmod 5$. Theorem 1.1 and the Schoenheim bound together imply that $\sigma(v, 5, 4) \leq b(v)$ where $b(v)$ is defined by

$$b(v) = \begin{cases} [v(v-1)/5] = \psi(v, 5, 4) & \text{when } v \not\equiv 3 \pmod 5 \\ [v(v-1)/5] - 1 = \psi(v, 5, 4) - 1 & \text{when } v \equiv 3 \pmod 5, \end{cases}$$

In the remainder of this paper we shall prove that this upper bound is achieved for all but two integers v. Specifically, we prove the following.

Theorem 1.3. *For all positive integers v we have $\sigma(v, 5, 4) = b(v)$ with two exceptions, namely $\sigma(7, 5, 4) = 7 = b(v) - 1$ and $\sigma(4, 5, 4) = 0 = b(v) - 2$.*

In Section 2 we discuss the structure of maximum packing designs, and in Section 3 we give the main recursive tools for proving Theorem 1.3. The final section is devoted to constructing maximum $(v, 5, 4)$ packing designs for small values of v and then giving an induction proof of the Theorem.

2. The structure of packing designs

Let (V, β) be a (v, k, λ) packing design, and for each 2-subset $e = \{x, y\}$ of V define $m(e)$ to be the number of blocks in β which contain e. Note that by the definition of a packing design we have $m(e) \leq \lambda$ for all e.

The *complement* of (V, β), denoted by $C(V, \beta)$, is defined to be the multigraph with vertex set V and edges e occurring with multiplicity $\lambda - m(e)$ for all e. The number of edges (counting multiplicities) in $C(V, \beta)$ is given by $\lambda\binom{v}{2} - |\beta|\binom{k}{2}$. The degree of the vertex x in $C(V, \beta)$ is $\lambda(v-1) - r_x(k-1)$ where r_x is the number of blocks containing x. These observations give rise to the following structure theorems.

Lemma 2.1. *Let (V, β) be a $(v, 5, 4)$ packing design with $|\beta| = b(v)$ then the degree of each vertex of $C(V, \beta)$ is divisible by* 4, *and the number of edges in the multigraph is* 0, 4, *or* 12 *when $v \bmod 5 \in \{0, 1\}$, $\{2, 4\}$ or $\{3\}$ respectively.*

The only multigraph with 4 edges and every vertex of degree divisible by 4 is the graph with four parallel edges connecting two vertices and $v-2$ isolated vertices. Therefore, when $v \equiv 2$ or $4 \pmod 5$ a $(v, 5, 4)$ packing design of cardinality $b(v)$ contains a pair of points which do not appear in any block, and all other pairs appear in precisely 4 blocks. This observation enables us to prove the following.

Lemma 2.2. $\sigma(7, 5, 4) < b(7) = 8$.

Proof. If $\sigma(7, 5, 4) = 8$ then there exists a packing design (V, β) on 7 points containing two points x, y which do not appear together in any block, so $|\beta| \geq r_x + r_y$. But each of the pairs $\{x, z\}$ with $z \neq y$ appear in four blocks. Hence $r_x = 5$. Similarly $r_y = 5$, but then we have $8 = |\beta| \geq r_x + r_y = 10$, a contradiction. □

The situation is more complicated when $v \equiv 3 \pmod 5$ since there are many multigraphs with 12 edges satisfying the degree constraint. A particularly useful multigraph of this type is the graph with $v-3$ isolated vertices and 3 vertices each connected to the other 2 by four parallel edges. If $C(V, \beta)$ is of this type then V contains 3 vertices x, y, z such that the pairs $\{x, y\}$, $\{x, z\}$, and $\{y, z\}$ appear in no blocks, and all other pairs appear in precisely 4 blocks.

These two configurations motivate the following definition. Let (V, β) be a (v, k, λ) packing design, and let H be a subset of V of cardinality h. We shall say that (V, β) is an *exact packing with a hole of size h* if no 2-subset of H appears in any block, and every other 2-subset of V appears in precisely λ blocks. We can now reformulate some of the comments above as follows.

Lemma 2.3. *Let $v \equiv 2$ or $4 \pmod 5$. An exact $(v, 5, 4)$ packing with a hole of size* 2 *exists if and only if $\sigma(v, 5, 4) = b(v)$.*

Lemma 2.4. *Let $v \equiv 3 \pmod 5$. If an exact $(v, 5, 4)$ packing with a hole of size* 3 *exists then $\sigma(v, 5, 4) = b(v)$.*

The converse of Lemma 2.4 is not true, since an argument similar to the proof of Lemma 2.2 shows that an exact $(8, 5, 4)$ packing with a hole of size 3 cannot exist, however the result $\sigma(8, 5, 4) = b(8)$ is proved in Section 4.

3. Recursive construction of packing designs

In order to describe our recursive constructions we need the notions of a transversal design, and a truncated transversal design. Let k, λ and w be positive integers. A *transversal design* $\mathrm{T}(k, \lambda, w)$ is a triple (V, β, γ) where V is a set of points with $|V| = kw$, and $\gamma = \{G_1, G_2, \ldots, G_k\}$ is a partition of V into k sets of size w. The parts, G_i, of the partition are called *groups*. The collection β consists of k-subsets of V, called *blocks*, with the following properties

1. $|B \cap G_j| = 1$ for all $B \in \beta$ and $G_j \in \gamma$.
2. Every 2-subset $\{x, y\}$ of V such that x and y belong to distinct groups is contained in exactly λ blocks.

It is well known that a $\mathrm{T}(k, 1, w)$ is equivalent to $k-2$ mutually orthogonal Latin squares of side w. An alternate description of a transversal design (V, β, γ) is that (V, β) is an exact (kw, k, λ) packing with k mutually disjoint holes of size w (the groups).

In the sequel we shall use the following existence theorems for transversal designs. The proofs of these results may be found in [3], [6], [7] and [10].

Theorem 3.1. *There exists a* $\mathrm{T}(6, 1, w)$ *for all positive integers* w *with the exception of* $w \in \{2, 3, 4, 6\}$ *and the possible exception of* $w \in \{10, 14, 18, 20, 22, 26, 28, 30, 34, 38, 42, 44, 52\}$.

Theorem 3.2. *There exists a* $\mathrm{T}(7, \lambda, w)$ *for all positive integers* w *and all integers* $\lambda \geq 2$.

We now give the definition of truncated transversal design. Let k, λ and w be positive integers, and let u be a nonnegative integer. A *truncated transversal design* $\mathrm{TT}(k, \lambda, w, u)$ is a triple (V, β, γ) where V is a set of points with $|V| = (k-1)w + u$, and $\gamma = \{G_1, G_2, \ldots, G_k\}$ is a partition of V into $k-1$ sets of size w, and one set G_k of size u. The parts G_i, of the partition are called *groups*. The collection β consists of k-subsets and $(k-1)$-subsets of V, called *blocks*, with the properties that

1. $|B \cap G_j| = 1$ for all $B \in \beta$ and $1 \leq j < k$.
2. $|B \cap G_k| = 1$ for all $B \in \beta$ such that $|B| = k$.
3. Every 2-subset $\{x, y\}$ of V such that x and y belong to distinct groups is contained in exactly λ blocks.

Clearly a $\mathrm{TT}(k, \lambda, w, 0)$ is equivalent to a $\mathrm{T}(k-1, \lambda, w)$. Furthermore, if $0 \leq u \leq w$ then one may construct a $\mathrm{TT}(k, \lambda, w, u)$ from a transversal design

$T(k, \lambda, w)$ by removing points from the last group, and from all the blocks which contain them. Thus we have the following existence results which are in the form most useful to us.

Theorem 3.3. *There exists a* $TT(6, 1, w, u)$ *for all integers* u *with* $0 \leq u \leq w$ *and for all positive integers* w *with the exception of* $w \in \{2, 3, 4, 6\}$ *and the possible exception of* $w \in \{10, 14, 18, 20, 22, 26, 28, 30, 34, 38, 42, 44, 52\}$.

Theorem 3.4. *There exists a* $T(5, 4, w)$ *for all positive integers* w.

We now give the recursive constructions used in the proof of our main theorem.

Theorem 3.5. *If there exists a* $TT(6, 1, w, u)$ *with* $w \equiv 0$ *or* $1 \pmod 5$, *and* $\sigma(u, 5, 4) = b(u)$ *then* $\sigma(5w + u, 5, 4) = b(5w + u)$.

Proof. On the blocks and the groups of size w of the truncated transversal design construct balanced incomplete block designs $B(v, 5, 4)$ with $v = 5$, 6 and w. On the group of size u construct a $(u, 5, 4)$ packing design with $b(u)$ blocks. This gives a $(5w + u, 5, 4)$ packing design with $b(5w + u)$ blocks. □

Theorem 3.6. *If there exists a* $TT(6, 1, w, u)$ *with* $w \equiv 0$ *or* $4 \pmod 5$ *and* $\sigma(u + 1, 5, 4) = b(u + 1)$ *then* $\sigma(5w + u + 1, 5, 4) = b(5w + u + 1)$.

Proof. Add a new point to each of the groups of the truncated transversal design. On the blocks and the new groups of size $w + 1$ construct balanced incomplete block designs $B(v, 5, 4)$ with $v = 5$, 6 and $w + 1$. On the new group of size $u + 1$ construct a $(u + 1, 5, 4)$ packing design with $b(u + 1)$ blocks. This gives a $(5w + u + 1, 5, 4)$ packing design with $b(5w + u + 1)$ blocks. □

Theorem 3.7. *If there exists a* $TT(6, 1, w, u)$ *with* $w \equiv 0$ *or* $2 \pmod 5$, *and* $\sigma(w + 2, 5, 4) = b(w + 2)$, *and* $\sigma(u + 2, 5, 4) = b(u + 2)$ *then* $\sigma(5w + u + 2, 5, 4) = b(5w + u + 2)$.

Proof. Add a two new points, h_1, h_2, to each of the groups of the truncated transversal design. On the blocks of the truncated transversal design construct balanced incomplete block designs $B(v, 5, 4)$ with $v = 5, 6$. On each of the new groups of size $w + 2$ construct an exact $(w + 2, 5, 4)$ packing design with a hole $H = \{h_1, h_2\}$ of size 2. This configuration exists by Lemma 2.3. On the new group of size $u + 2$ construct a $(u + 2, 5, 4)$ packing design with $b(u + 2)$ blocks. This gives a $(5w + u + 2, 5, 4)$ packing design with $b(5w + u + 2)$ blocks. □

Theorem 3.8. *If* $w \equiv 0$ *or* $2 \pmod 5$, *and* $\sigma(w + 2, 5, 4) = b(w + 2)$, *then* $\sigma(5w + 2, 5, 4) = b(5w + 2)$.

Proof. By Theorem 3.4, we can construct a transversal design T(5, 4, w). Add a two new points, h_1, h_2, to each of the groups of this transversal design. On each of the new groups of size $w+2$ construct an exact $(w+2, 5, 4)$ packing design with a hole $H=\{h_1, h_2\}$ of size 2. The blocks of these designs together with the blocks of the transversal design are a $(5w+2, 5, 4)$ packing design with $b(5w+2)$ blocks. □

Theorem 3.9. *If* $w \equiv 0 \pmod 5$, *and there exists an exact* $(w+3, 4, 5)$ *packing design with a hole of size* 3, *then there exists an exact* $(5w+3, 5, 4)$ *packing design with a hold of size* 3, *and hence* $\sigma(5w+3, 5, 4)=b(5w+3)$.

The proof of this theorem is exactly analogous to the proof of Theorem 3.8, and is omitted.

4. Proof of the main theorem

Before giving an induction proof of Theorem 1.3, we require the direct construction of some packing designs with small values of v. It is trivial to observe that $\sigma(v, k, \lambda)=0$ for all $v<k$, and hence $\sigma(2, 5, 4)=\sigma(3, 5, 4)=\sigma(4, 5, 4)=0$. The next cases of interest are $v=7, 8$, these are covered in the next lemma.

Lemma 4.1. $\sigma(7, 5, 4)=7=b(7)-1$ *and* $\sigma(8, 5, 4)=10=b(8)$.

Proof. By Lemma 2.2 we have $\sigma(7, 5, 4)\leq 7$, and the following is a $(7, 5, 4)$ packing design with 7 blocks. Let $V=Z_7$ and let β be the orbit of the base block $\{0, 1, 2, 4, 5\}$ under the action of the cyclic group generated by the permutation of V which sends $i \to i+1 \pmod 7$.

For $v=8$ let $V=Z_7 \cup \{a\}$ and let β be the orbit of the base block $\{a, 0, 1, 2, 4\}$ under the action of the cyclic group generated by the permutation of V which fixes a and sends $i \to i+1 \pmod 7$, together with the three blocks $\{i, i+1, i+2, i+4, i+5\}$ with $i=0, 1, 2$ (addition modulo 7). □

We now give a construction using finite fields.

Theorem 4.2. *Let n be a positive integer such that* $3n+1$ *is a prime power. Then there exists an exact* $(4n+1, 5, 4)$ *packing design with a hole of size n.*

Proof. Let $V=\mathrm{GF}(3n+1) \cup \{h_0, h_1, \ldots, h_{n-1}\}$, and let x be a primitive element of GF$(3n+1)$. Now construct the block set β as follows.

$$\beta=\{\{h_i, y, y+x^i, y+x^{n+i}, y+x^{2n+i}\}: 0 \leq i<n, y \in \mathrm{GF}(3n+1)\}.$$

The proof that this is indeed an exact $(4n+1, 5, 4)$ packing design with a hole of size n is by standard difference set techniques. □

We now give a table (*Fig. 1*) describing the construction of some exact $(v, 5, 4)$ packing designs with a hole of size n. In general, the construction is as follows. Let $V = Z_{v-n} \cup H_n$ where $H_n = \{h_0, h_1, \ldots, h_{n-1}\}$ is the hole. The blocks are constructed by taking the orbits of the tabulated base blocks under the action of the cyclic group generated by the permutation which fixes the elements of H_n and sends $i \to i+1 \pmod{v-n}$ for each $i \in Z_{v-n}$. The notation $\times m$ following a base block indicates that the entire orbit is to be taken m times. Except where otherwise stated the length of each orbit is $v - n$. The exceptions to this construction scheme are when $v = 12$, 17 and 59.

When $v = 59$ it is more convenient to use the point set $V = Z_{47} \cup H_{12}^*$ where $H_{12}^* = \{h_j : j = 1, 2, 3, 4, 6, 7, 8, 9, 21, 22, 23, 24\}$ is the hole of size 12.

When $v = 17$ we use the point set $V = Z_{12} \cup H_2 \cup \{x, y, z\}$ and the blocks are constructed by taking the orbits of the tabulated base blocks under the action of the cyclic group (of order 12) generated by the permutation $(h_0\, h_1)(x\, y\, z)(0\, 1\, 2 \cdots 1\, 1)$.

When $v = 12$ we use the point set $V = Z_{10} \cup H_2$ and the blocks are constructed by taking the orbits of the tabulated base blocks under the action of the cyclic group (of order 5) generated by the permutation $(h_0)(h_1)(0\,2\,4\,6\,8)(1\,3\,5\,7\,9)$.

The following lemma is our main reason for constructing exact packings with holes.

Lemma 4.3. *If an exact* $(v, 5, 4)$ *packing with a hole of size* h *exists and* $\sigma(h, 5, 4) = b(h)$ *then* $\sigma(v, 5, 4) = b(v)$.

Proof. Form the blocks of an $(h, 5, 4)$ packing with $b(h)$ blocks on the points of the hole. Adding the blocks of the exact packing design gives a $(v, 5, 4)$ packing design with $b(v)$ blocks. □

Corollary 4.4. *For all the values of* v *appearing in Fig. 1, and for* $v = 8, 9, 33, 49$ *we have* $\sigma(v, 5, 4) = b(v)$.

Proof. For $v = 9$, 33, 49 Theorem 4.2 proves the existence of an exact $(v, 5, 4)$ packing design with a hole of size 2, 8, and 12 respectively. Thus the corollary is true by the constructions in *Fig. 1*, and Lemmas 2.3, 2.4, 4.1, and 4.3. □

We now begin the proof of Theorem 1.3, which is restated below for the reader's convenience.

Theorem 1.3. *For all positive integers* v *we have* $\sigma(v, 5, 4) = b(v)$ *with two exceptions, namely* $\sigma(7, 5, 4) = 7 = b(v) - 1$ *and* $\sigma(4, 5, 4) = 0 = b(v) - 2$, *where*

$$b(v) = \begin{cases} [v(v-1)/5] = \psi(v, 5, 4) & \text{when } v \not\equiv 3 \pmod 5 \\ [v(v-1)/5] - 1 = \psi(v, 5, 4) - 1 & \text{when } v \equiv 3 \pmod 5. \end{cases}$$

v	n	Point Set	Base Blocks
12	2	$Z_{10} \cup H_2$	$\{h_0, 0, 1, 4, 5\}$ $\{h_0, 0, 4, 7, 9\}$ $\{h_1, 0, 1, 3, 9\}$ $\{h_1, 0, 2, 4, 7\}$ $\{0, 1, 2, 5, 9\}$ $\{0, 2, 4, 6, 8\}$ (orbit length 1)
13	3	$Z_{10} \cup H_3$	$\{h_0, 0, 1, 3, 8\}$ $\{h_1, 0, 1, 5, 9\}$ $\{h_2, 0, 1, 3, 7\}$
14	2	$Z_{12} \cup H_2$	$\{h_0, 0, 2, 4, 5\}$ $\{h_1, 0, 2, 5, 6\}$ $\{0, 1, 4, 5, 7\}$
17	2	$Z_{12} \cup H_2 \cup \{x, y, z,\}$	$\{h_0, x, 0, 1, 11\}$ $\{h_1, x, 0, 5, 7\}$ $\{y, z, 0, 2, 10\}$ $\{0, 3, 4, 8, 9\}$ $\{h_0, 0, 3, 6, 9\}$ (orbit length 6)
18	3	$Z_{15} \cup H_3$	$\{h_0, 0, 1, 3, 8\}$ $\{h_1, 0, 1, 10, 12\}$ $\{h_2, 0, 1, 5, 7\}$ $\{0, 1, 3, 7, 12\}$
19	2	$Z_{17} \cup H_2$	$\{h_0, 0, 3, 4, 9\}$ $\{h_1, 0, 2, 5, 9\}$ $\{0, 1, 3, 5, 11\}$ $\{0, 1, 5, 7, 8\}$
22	2	$Z_{20} \cup H_2$	$\{h_0, 0, 2, 5, 15\}$ $\{h_1, 0, 5, 11, 14\}$ $\{0, 1, 3, 4, 10\}$ $\{0, 1, 2, 7, 9\}$ $\{0, 4, 8, 12, 16\} \times 3$ (orbit length 4)
23	3	$Z_{20} \cup H_3$	$\{h_0, 0, 6, 7, 9\}$ $\{h_1, 0, 6, 8, 16\}$ $\{h_2, 0, 1, 7, 10\}$ $\{0, 1, 3, 5, 8\}$ $\{0, 6, 7, 11, 15\}$
24	2	$Z_{22} \cup H_2$	$\{h_0, 0, 9, 15, 17\}$ $\{h_1, 0, 6, 18, 19\}$ $\{0, 2, 7, 10, 11\} \times 2$ $\{0, 4, 6, 7, 12\}$
29	2	$Z_{27} \cup H_2$	$\{h_0, 0, 10, 11, 13\}$ $\{h_1, 0, 12, 17, 21\}$ $\{0, 1, 3, 7, 16\}$ $\{0, 3, 8, 14, 18\}$ $\{0, 5, 7, 8, 15\}$ $\{0, 2, 8, 9, 13\}$
32	2	$Z_{30} \cup H_2$	$\{h_0, 0, 4, 5, 14\}$ $\{h_1, 0, 7, 13, 15\}$ $\{0, 2, 7, 16, 27\}$ $\{0, 4, 12, 15, 29\}$ $\{0, 3, 4, 11, 13\} \times 2$ $\{0, 6, 12, 18, 24\} \times 3$ (orbit length 6)
34	2	$Z_{32} \cup H_2$	$\{h_0, 0, 1, 4, 13\}$ $\{h_1, 0, 1, 4, 13\}$ $\{0, 2, 7, 18, 24\} \times 2$ $\{0, 1, 2, 4, 24\}$ $\{0, 3, 8, 15, 21\}$ $\{0, 4, 9, 15, 22\}$
39	2	$Z_{37} \cup H_2$	$\{h_0, 0, 1, 13, 30\}$ $\{h_1, 0, 4, 10, 19\}$ $\{0, 3, 12, 16, 23\}$ $\{0, 1, 6, 16, 35\}$ $\{0, 14, 29, 34, 35\}$ $\{0, 14, 24, 26, 33\}$ $\{0, 19, 26, 32, 34\}$ $\{0, 4, 16, 27, 36\}$
42	9	$Z_{33} \cup H_9$	$\{h_0, 0, 6, 14, 15\}$ $\{h_1, 0, 2, 10, 23\}$ $\{h_2, 0, 1, 5, 16\}$ $\{h_3, 0, 14, 20, 21\}$ $\{h_4, 0, 3, 6, 17\}$ $\{h_5, 0, 8, 10, 15\}$ $\{h_6, 0, 4, 13, 15\}$ $\{h_7, 0, 3, 17, 27\}$ $\{h_8, 0, 7, 8, 12\}$ $\{0, 3, 5, 12, 16\}$
59	12	$Z_{47} \cup H_{12}^{\star}$	$\{h_i, 0, i, 6i, 22i\}$ $i \in \{1,6,21\}$ $\{h_{i+1}, 0, i, 10i, 34i\}$ $i \in \{1,6,21\}$ $\{h_{i+2}, 0, 4i, 12i, 40i\}$ $i \in \{1,6,21\}$ $\{h_{i+3}, 0, 3i, 18i, 20i\}$ $i \in \{1,6,21\}$ $\{0, 4, 7, 22, 34\}$ $\{0, 8, 31, 36, 45\}$
67	14	$Z_{53} \cup H_{14}$	$\{h_i, 2^i, 2^{13+i}, 2^{26+i}, 2^{39+i}\}$ $i = 0,1,\ldots,12$ $\{h_{13}, 0, 2, 8, 15\}$ $\{0, 1, 22, 25, 42\}$ $\{0, 4, 14, 30, 48\}$

Fig. 1. Exact $(v, 5, 4)$ packing designs with a hole of size n.

Proof. The theorem is true for $v \equiv 0$ or $1 \pmod 5$ by Theorem 1.2. The theorem is also true for $v \leqslant 24$, by Lemma 4.1 and Corollary 4.4. For $v \equiv 2$, 3 and $4 \pmod 5$ with $v \geqslant 25$ we consider four cases.

Case 1. $v = 2, 3, 9 \pmod{25}$.
In this case $v = 5w + u$ where $w \equiv 0 \pmod 5$, $w \geqslant 5$, and $u \in \{2, 3, 9\}$. By Theorem 3.3 there exists a TT(6, 1, w, u) for all the relevant pairs (w, u) with the exception of $w \in \{10, 20, 30\}$ and $(w, u) = (5, 9)$. So for $v \neq 34$, 52, 53, 59, 102, 103, 109, 152, 153, 159, apply Theorem 3.5 to give the result. For $v = 34$, 59 use Corollary 4.4. For $v = 52$, 102, 152 we can apply Theorem 3.8 with $w = 10$, 20, 30 since a T(5, 4, w) exists by Theorem 3.4, and $\sigma(w+2, 5, 4) = b(w+2)$ by Corollary 4.4. For $v = 53$, 103 we can apply Theorem 3.9 with $w = 10$, 20 since a T(5, 4, w) exists by Theorem 3.4, and an exact $(w+3, 5, 4)$ packing with a hole of size 3 exists by *Fig. 1.* For the remaining values of v see *Fig. 2.*

Case 2. $v \equiv 7, 8, 14, 17, 18 \pmod{25}$.
In this case $v = 5w + u$ where $w \equiv 1 \pmod 5$, $w \geqslant 6$, and $u \in \{2, 3, 9, 12, 13\}$. By Theorem 3.3 there exists a TT(6, 1, w, u) for all the relevant pairs (w, u) with the

v	w	u	Thm.	v	w	u	Thm.	v	w	u	Thm.
43	7	6	3.7	133	24	12	3.6	213	40	13	3.5
68	12	6	3.7	139	24	18	3.6	219	41	14	3.5
72	12	10	3.7	142	24	21	3.6	222	41	17	3.5
73	12	11	3.7	143	24	22	3.6	223	41	18	3.5
109	19	13	3.6	153	27	16	3.7	224	41	19	3.5
112	19	16	3.6	159	27	22	3.7	229	41	22	3.5
113	19	17	3.6	172	31	17	3.5	262	50	12	3.5
119	21	14	3.5	173	31	18	3.5	263	50	13	3.5
124	21	19	3.5	179	31	22	3.5	269	50	19	3.5
132	24	11	3.6	212	40	12	3.5	274	50	24	3.5

Fig. 2. Exceptional constructions for $(v, 5, 4)$ packing designs.

exception of $w \in \{6, 26\}$ and $(w, u) = (11, 12)$, $(11, 13)$. So for $v \neq 32$, 33, 39, 42, 43, 67, 132, 133, 139, 142, 143, apply Theorem 3.5 to give the result. For $v = 32$, 33, 39, 42, 67 use Corollary 4.4. For the remaining values of v see *Fig. 2*.

Case 3. $v \equiv 4, 22, 23 \pmod{25}$.
In this case $v = 5w + u + 1$ where $w \equiv 4 \pmod 5$ and $u \in \{1, 2, 8\}$. By Theorem 3.3 there exists a TT(6, 1, w, u) for all the relevant pairs (w, u) with the exception of $w \in \{4, 14, 34, 44\}$. So for $v \neq 29$, 72, 73, 79, 172, 173, 179, 222, 223, 229, apply Theorem 3.6 to give the result. For $v = 29$ use Corollary 4.4. For $v = 79$ let $V = (Z_7 \times Z_{11}) \cup H_2$ and let (Z_{11}, β^*) be a B(11, 5, 4) design, which exists by Theorem 1.2. For each block $B_i \in B^*$ construct a T(5, 1, 7) with point set $Z_7 \times B_i$, groups $Z_7 \times \{j\}$, $j \in B_i$ and block set β_i. (The transversal designs exist by Theorem 3.3). Now let β be the union of all the β_i together with the blocks of exact (9, 5, 4) packings with hole H_2 on the point sets $(Z_7 \times \{j\}) \cup H_2$, $j \in Z_{11}$. For the remaining values of v see *Fig. 2*.

Case 4. $v \equiv 12, 13, 19, 24 \pmod{25}$.
In this case $v = 5w + u + 2$ where $w = 2 \pmod 5$ and $u \in \{0, 1, 7, 12\}$. By Theorem 3.3, there exists a TT(6, 1, w, u) for all the relevant pairs (w, u) with the exception of $w \in \{22, 42, 52\}$ and $(w, u) = (7, 12)$. So for $v \neq 49$, 112, 113, 119, 124, 212, 213, 219, 224, 262, 263, 269, 274 apply Theorem 3.7 to give the result. For $v = 49$ use Corollary 4.4 For the remaining values of v see *Fig. 2*. This completes the proof of the theorem. □

Careful inspection of the above proof yields the following.

Corollary 4.3. *For all $v \equiv 3 \pmod 5$ with $v \neq 8$ there exists an exact $(v, 5, 4)$ packing with a hole of size* 3, *with the possible exceptions of $v = 43$, 68.*

Proof. The non-existence of an exact (8, 5, 4) packing with a hole of size 3 follows from a similar argument to the proof of Lemma 2.2. In the proof of Theorem 1.3 we have avoided using holes of size 8, except when $v = 33$, 43 and 68. For $v = 33$ we have the following construction. Let $V = Z_{30} \cup H_3$ and construct β as in *Fig. 1*, using the base blocks $\{h_0, 0, 1, 4, 6\}\{h_1, 0, 2, 4, 6\}$ $\{h_2, 0, 7, 15, 22\} \times 2$ (orbit length 15) $\{0, 1, 10, 12, 17\} \times 2\{0, 3, 11, 17, 21\} \times 2$. □

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